

# Greenberg-Hastings dynamics on a small-world network: the effect of disorder on the collective extinct-active transition

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## Abstract

We present a numerical study of a reaction-diffusion model on a small-world network. We characterize the model's average activity  $F_T$  after  $T$  time steps and the transition from a collective (global) extinct state to an active state in parameter space. We provide an explicit relation between the parameters of our model at the frontier between these states. A collective active state can be associated to a global epidemic spread, or to a persistent neuronal activity. We found that  $F_T$  does not depend on disorder in the network if the transmission rate  $r$  or the average coordination number  $K$  are large enough. The collective extinct-active transition can be induced by changing two parameters associated to the network:  $K$  and the disorder parameter  $p$  (which controls the variance of  $K$ ). We can also induce the transition by changing  $r$ , which controls the threshold size in the dynamics. In order to operate at the transition the parameters of the model must satisfy the relation  $rK = a_p$ , where  $a_p$  as a function of  $p/(1-p)$  is a stretched exponential function. Our results are relevant for systems that operate *at* the transition in order to increase its dynamic range and/or to operate under optimal information-processing conditions. We discuss how glassy behaviour appears within our model.

**Keywords:** Complex systems, Cellular automata, Phase transitions

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Many problems in Science can be cast in terms of dynamics on networks: social phenomena [1, 2, 3], epidemic spread [4], food webs [5] and ecosystem's diversity [6], brain activity [7, 8, 9, 10, 11, 12], granular materials [13, 14, 15] and, in general, complex systems [16]. Among the most studied models, the small-world network model of Watts and Strogatz (WS) [17, 18, 10] can be tuned to interpolate between a regular and a random network, a very attractive property that allows us to explore the consequences of network disorder on dynamics. In this work we consider a stochastic reaction-diffusion cellular automata model on a small-world network and study its average activity after  $T$  time steps and its collective extinct-active transition. For the first time, we provide an explicit relation between the parameters of the model for the system to operate *at* the transition, and disorder enters in this relation as a stretched exponential function.

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In the WS model we start from an ordered ring in which each node is connected to its  $K$  nearest neighbours, and then we randomly rewire each connexion with probability  $p$ . Since we only rewire and we don't create new connexions, the average coordination number is still  $K$ , but the distribution of  $K$  gets broader as we increase  $p$  [18]. Two quantities typically used to characterize a network are the mean minimal distance between two nodes  $L$  and the clustering coefficient  $C$ . For each node, we can measure how many of its neighbours are connected between them.  $C$  is defined as the actual number of connexions between a node's neighbours over the maximum number of possible connexions between neighbours, averaged over all nodes. Hence, in a social network  $C$  could measure up to what extent someone's friends are friends between them. In the WS network there is a range of values of  $p$  for which  $C$  is still high and  $L$  is small, a signature of a small-world scenario. The case  $p = 0$  corresponds to an ordered network, with high  $C$  and  $L$ . The case  $p = 1$  corresponds to a random network, with small  $C$  and  $L$  [18, 17].

A very generic reaction-diffusion model is the Greenberg-Hastings model (GH). This model has been used to model the Belousov chemical reaction, biological systems, epidemic spread, collective behaviour of heart cells and neuronal activity [19, 20, 21, 22, 23, 24, 25, 7]. Our stochastic version of this three state model is as follows:

- If a cell is in the *excited* state at time  $t$  then it is in the *passive* state at time  $t + 1$ .
- If a cell is in the *passive* state at time  $t$  then it is in the *susceptible* state at time  $t + 1$ .
- If a given cell is in the *susceptible* state at time  $t$ , and at least one of its neighbours is in the *excited* state at time  $t$ , then the given cell is in the *excited* state at time  $t + 1$  with probability  $r$ , otherwise the given cell remains in the *susceptible* state.

The GH is a model for excitable media ([26], see fig. 1 in [32]) whose threshold is controlled by the parameter  $r$ . We call the parameter  $r$  the transmission (or infection) probability. If *excited* means no-healthy, then I become infected if at least one of my neighbours is infected, but with probability  $r$ . In this context  $r$  would be a measure of the average state of the immune system of the population, with a smaller  $r$  implying more resistance to become ill. If we are in a social context,  $r$  could be associated to the confidence of the social agents on an specific action, with greater  $r$  implying more confidence. For neuronal activity or chemical reactions,  $r$  can be associated to a potential barrier, with a smaller  $r$  implying a greater potential barrier (a greater threshold). In a raw model for dense granular flows, the states of the GH model could be associated to fluid or solid type of contacts between grains [27], with the *excited* state associated to fluid contacts, the *passive* state associated to contacts with static friction and with a large time of contact [27] or with a small mobilization of friction [15], and the *susceptible* state associated to contacts with static friction and with a short time of contact or with a large mobilization of friction. Versions of the GH model have been implemented on regular networks [28, 22, 24], scale free networks [25], on a small-world network to study spiral waves [29] and on the human connectome [7].

It is of particular interest under what conditions the system, for a given initial condition, evolves towards a global active state, in which a finite fraction of the nodes remains active [22]. A collective active state can be associated to a global epidemic spread, or to a persistent neuronal activity. For random initial conditions, we have found transitions

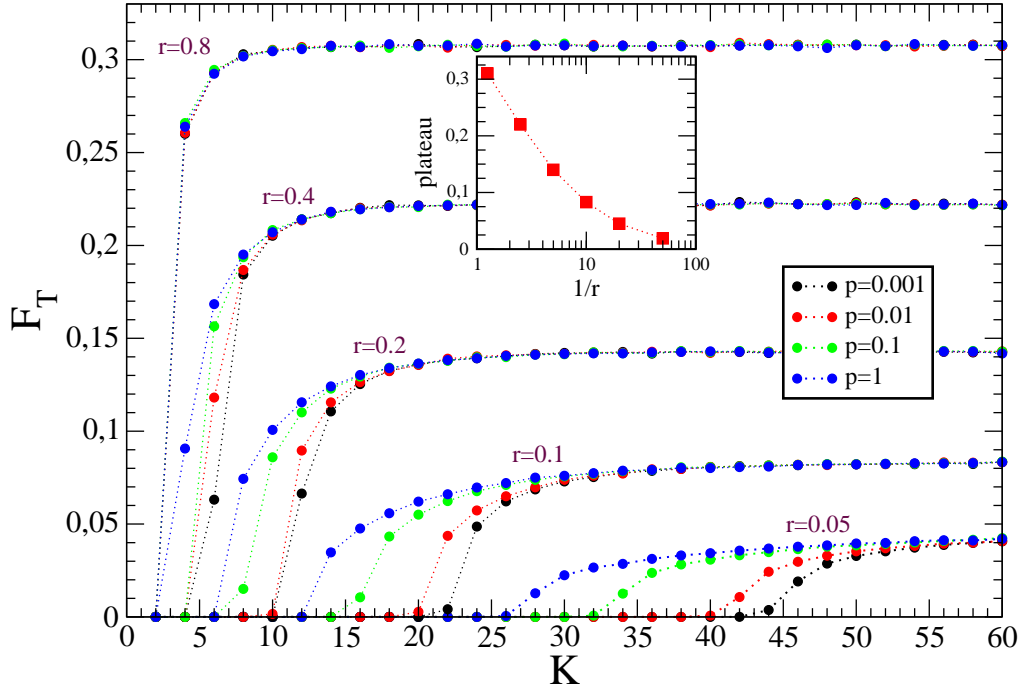


Figure 1: Average activity  $F_T$  as a function of the average coordination number  $K$ , for several values of  $p$  and  $r$ . In the inset is shown the plateau of  $F_T$ , obtained for large  $K$ , as a function of  $1/r$ . For  $r = 0.05$  and  $p = 0.1$ , for example, we have a collective active state for  $K > 32$ . If  $r \rightarrow 1$  or if  $K$  is large enough  $F_T$  becomes independent of the disorder parameter  $p$ . The number of nodes is  $N = 1000$  and we show the average result for 100 realizations for each set of parameters. For this figure, we looked for activity in the system after  $T = 1000$  time steps.

to a collective active state as we vary any of the three parameters in the GHWS model: the transition can be induced by increasing the average coordination number  $K$ , by increasing the disorder in the network  $p$  or by increasing the transmission probability  $r$ . Based on our numerical results, we provide for the first time an explicit relation between the parameters of the model for the system to operate *at* the transition, and disorder enters this relation as a stretched exponential function. Stretched exponentials are usually found in glasses and disordered systems [33, 34, 35]. As far as we know, for this model (GHWS) [29] we are presenting a novel way to consider the influence of disorder on the extinct-active frontier. In reference [7] the network was the human connectome and only a threshold was varied in order to tune the transition.

Our system consists of  $N = 1000$  nodes. Initially, each node is randomly assigned, with equal probability, to one of the three states of the model. We call  $F$  the average number of nodes in the excited state. The behaviour of the GHWS model is summarized in figure 1, on which we show the average activity after  $T = 1000$  time steps  $F_T(K, p, r)$ . We have a collective active state if after  $T$  time steps (see below)  $F_T > 0$ . We have a collective extinct state if after  $T$  time steps  $F_T = 0$ . For  $K > K_c$  we have a collective active state, and from this figure we see that  $K_c = K_c(r, p)$ . It can be observed that for  $r \rightarrow 1$   $F_T$  becomes independent of the disorder parameter  $p$ . Also, for any given transmission probability  $r$ ,  $F_T$  becomes independent of  $p$  if  $K$  is large enough: it tends to a plateau whose dependence on  $1/r$  is shown in the inset.

For  $r = 0.05$ , we show in figure 2a zones of global active states in parameter space. We have found, as previously reported [30], that for  $K < K_c$  the relaxation time of decay to zero activity increases largely as we approach a transition to a collective active state. Because of this, we looked for activity after  $T$  time steps, and by a *time step* we mean an update of the whole network. Thus, we are characterizing the evolution in time of the frontier, and the limit  $T \rightarrow \infty$  corresponds to the usual meaning of the frontier. For a given  $r$ , we can induce a transition to a collective active state by increasing the disorder parameter  $p$  or by increasing the average coordination number  $K$ .

In order to characterize the frontier between active and extinct global states in parameter space we show in figures 2b and 2c the critical transmission probability  $r_c$  as a function of  $1/K$ , for disorder spanning four orders of magnitude in  $p$ . Our results can be summarized in the following simple relation:

$$r_c = \frac{a_p}{K}. \quad (1)$$

This scaling of  $r_c$  with  $K$  was obtained by Berry and Fates [22] in a mean field approximation of a similar GH model, where the crucial ingredient was to obtain an approximation to the conditional probability that in the neighbourhood of a given node there is at least one node in the excited state, given that the considered node is in the susceptible state. The scaling  $r_c \sim K^{-1}$  was verified by numerical simulations on a regular network by considering different types of neighbourhoods, obtaining  $a_p \approx 2$  (the exact mean field result was  $a_p = 1$ ) [22]; this result was robust against the inclusion of defects in the regular network. In figure 2d we show  $a_p$  as a function of  $p$ , and we conclude from this figure that disorder in the network reduces  $r_c$ . Since we have a collective active state for  $r > r_c$ , within the GHWS model disorder favours a collective active state.

In figure 3 we see a detailed version of figure 2d. Our numerical results for  $a_p$  are

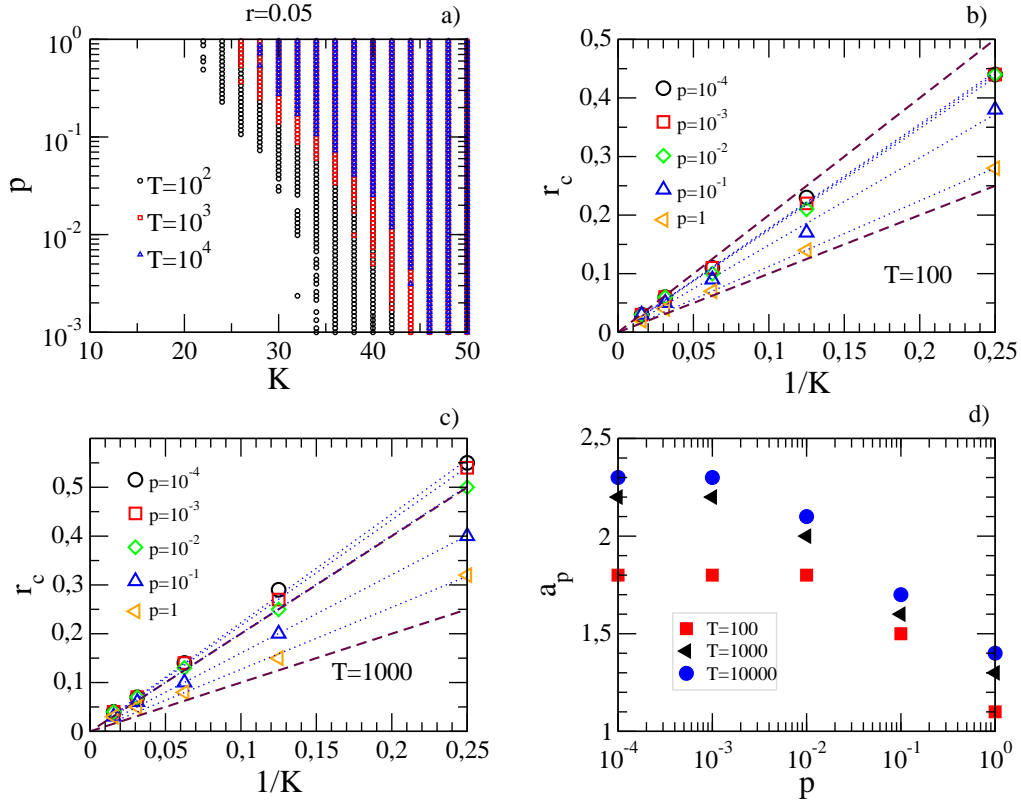


Figure 2: a) Zones of collective active states for  $r = 0.05$  after  $T$  time steps. b) Critical transmission probability  $r_c$  as a function of  $1/K$ , estimated after  $T = 100$  time steps for different  $p$ . c) Same as in b) but with  $T = 1000$ . As in b), the dashed lines corresponds to the relations  $r_c = 1/K$  and  $r_c = 2/K$ . Our system is a network with  $N = 1000$  nodes and we averaged over  $m = 100$  realizations. Dotted lines, here and in b), are best fits to the relation  $r_c = a_p/K$ . We don't considered the case  $r = 1$  (the deterministic GH model) for which we have a collective active state for  $K = 2$ , for any  $p$ . d) The slope  $a_p$  as a function of the disorder parameter  $p$ , obtained from c) and b).

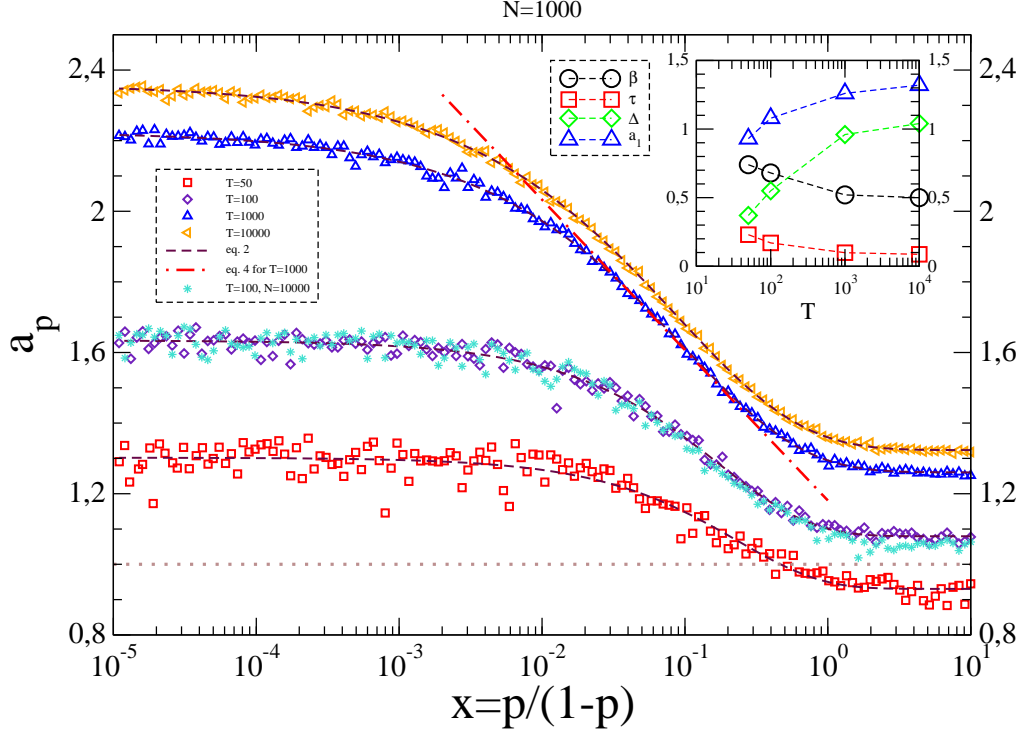


Figure 3: This figure is a detailed version of figure 2d. In order to obtain each point in this graph we considered for each  $p$  five values of  $K$ :  $K = 4, 8, 16, 32, 64$ , with  $m$  realizations for each  $K$ . We then adjusted the best line to equation (1). For  $T = 50$  and  $T = 100$ , we used  $m = 1000$ ; for  $T = 1000$  we used  $m = 100$ ; for  $T = 10000$  we used  $m = 30$ . Dashed lines are best fits to stretched exponentials of the form given by equation 2. In the inset we show the four parameters of eq. 2 as a function of  $T$ . The dot-dashed line is equation 4 for  $T = 1000$ . We show, in stars, the results for  $N = 10000$  and  $T = 100$ , with no major differences with the case  $N = 1000$ ; certainly  $L$  depends on  $N$  [18] but  $L$  appears to affect only the time required to propagate the activity to the whole system [17]. The dotted line is the mean field result  $a_p = 1$  [22]. Recently (v6), I have been aware that if we average in a different way we do have a  $N$  dependence: if we do only one realization for each value of  $K$  (see the beginning of this caption), obtaining  $a_p$  and then average several realizations we do have a  $N$  dependence in the results. This fact will be considered elsewhere. For  $N = 1000$ , the results presented here do not change qualitatively.

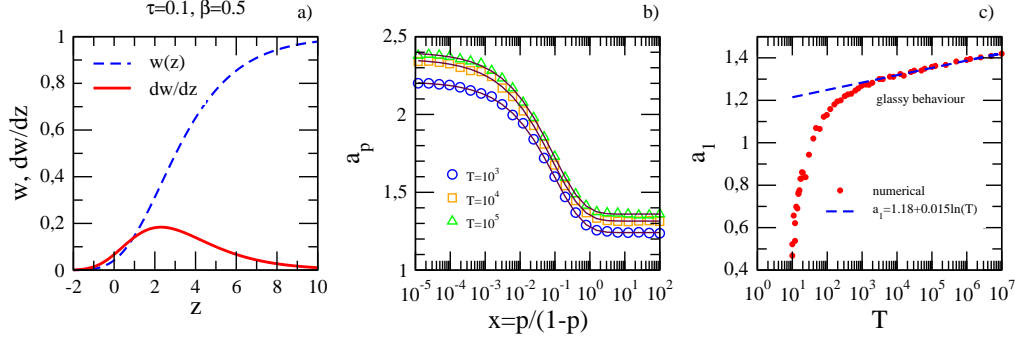


Figure 4: a) Behaviour of  $w \equiv (a_p - a_1)/\Delta$ , and it's derivative, with  $\chi = \Delta \partial w / \partial z$ . b)  $a_p$  for  $T = 10^3$ ,  $T = 10^4$  and  $T = 10^5$  (different run from previous figure). Dashed lines are best fits to stretched exponentials of the form given by eq. (2). At  $T = 10^4$  the frontier no longer changes it's shape, all the parameters of the stretched exponential no longer changes with time, except for  $a_1$ . For  $rK < a_p$  we have an extinct state. c) The frontier keeps moving since  $a_1$  keeps increasing (as  $a_1 \sim 0.015 \ln(T)$  for  $T > 2000$ ). We could say that for  $T > 2000$  the frontier enters a glassy-like regime.

consistent with a stretched exponential of the form:

$$a_p = \Delta \exp[-(x/\tau)^\beta] + a_1 \quad (2)$$

with  $x = p/(1-p)$ . Stretched exponentials are usually found in relaxation of glasses and disordered systems [33, 34, 35, 36, 37]. In the inset of figure 3 we show the evolution of the four parameters of equation 2.

In terms of  $z \equiv \ln(1/x)$ , there is a maximum in the quantity  $\chi(z) \equiv \partial a_p / \partial z$  at  $z_c = \ln(1/\tau)$ , with the maximum given by  $\chi_m \equiv \chi(z_c)$ :

$$\chi_m = \frac{\Delta \beta}{e} \quad (3)$$

The quantity  $\chi$  measures the sensitivity of the extinct-active frontier to changes in disorder (through the variable  $z$ ), going to zero for very ordered or very disordered networks. In terms of  $\mu \equiv e^{-z}/\tau$ , the order parameter  $w \equiv (a_p - a_1)/\Delta = \exp(-\mu^\beta)$  and  $\chi(z) = \Delta \partial w / \partial z$ , we have:  $\chi(z)/\chi_m = e\mu w$  (see figure 4a). If we operate at  $z_c$  we can communicate more easily through collective states when considering, for example, a network of networks [38] in which each unit is a GHWS one: at the frontier and at  $z_c$  small changes in disorder can allow the system to switch between an extinct and an active collective state more easily.

If we expand around  $z_c$ , we obtain the approximate relation:

$$a_p - a_{pc} \approx \chi_m (z - z_c) \quad (4)$$

with  $a_{pc} = \Delta/e + a_1$ . In figure 3 we show eq. 4 for  $T = 1000$ .

By monitoring the activity  $F$  as a function of  $T$  we have found that  $a_p$  is well described by a stretched exponential function in the variable  $x = p/(1-p)$ , for all  $T$ . Since at the extinct-active frontier  $rK = a_p$  (eq. 1), we can interpret  $a_p$  as the average minimal number of *effective* neighbours to become active. In eq. 2 we have defined  $\Delta$  as the

difference between  $a_0$  ( $a_p$  when  $p \rightarrow 0$ ) and  $a_1$  ( $a_p$  when  $p \rightarrow 1$ ), see the inset of figure 3. What happens, for example, if we consider an ensemble of systems such that  $rK \approx 1.63$ , in which each member of the ensemble can have any value of  $p$ ?. We see in figure 3 that  $a_0 \approx 1.63$  for  $T = 100$ . Thus, up to  $T = 100$  we'll have ordered and disordered members of the ensemble that are still active, but in the long run only the more disordered members of the ensemble ( $x > 0.2$ , approx., see fig. 3) will stay active. If we consider an ensemble of systems with  $rK < a_1$ , then in the long run all the members of the ensemble will go extinct. On the other hand, if we consider an ensemble of systems with  $rK > a_0$  then all the members of the ensemble will stay active in the long run.

In figure 4b we can see that the frontier no longer changes its shape for  $T > 10^4$ , only  $a_1$  keeps increasing with time, as can be seen in figure 4c. With the relation  $a_1 = 1.18 + 0.015 \ln(T)$  being satisfied for  $T > 2000$  (see fig. 4c), we have that for an ensemble with  $rK \approx 2.4$ , at  $T_0 = 10^5$  the more ordered members of the ensemble starts to go extinct (see fig. 4b). For this ensemble, if we denote  $T_1$  as the time when the more disordered members of the ensemble will go extinct we have that  $T_1/T_0 = e^{\Delta/0.015} \approx e^{67} \approx 10^{29}$ . This is reminiscent of glassy behaviour.

Several generalizations can be introduced in the model implemented here. One of them would be to consider a weighted network, with weights that can be correlated or not to the local coordination number [3]. In order to consider different passive time scales, a possible generalization of our GH model would be to introduce  $M$  time steps in the passive (refractory) state before becoming susceptible. Regarding this possibility, in reference [22] was found that, for a regular network, the critical transmission probability depends weakly on  $M$ , and that at the active-extinct transition the decay in time of the average activity  $F$  follows a power law, with an exponent that does not depends on  $M$ : its universality class remains directed percolation.

We have characterized the activity of a GHWS model, and we found that the average activity does not depends on disorder in the network if the transmission rate  $r$  or the average coordination number  $K$  are large enough. We have found that a collective extinct-active transition in a GHWS model can be induced by increasing  $r$ , a parameter that controls the threshold size in the system's dynamics. Also, the transition can be induced by increasing the disorder parameter  $p$  and by increasing  $K$ . Our results are relevant to systems that need to operate at the extinct-active transition, in order to increase its dynamic range and/or to operate under optimal information-processing conditions [31]. Remarkably, in reference [7] it was found that in order to reproduce the patterns of neuronal activity observed in experiments, the threshold of a stochastic GH model implemented on the human connectome has to be tuned for the system to operate at the the extinct-active transition [39]. Within the GHWS model studied here, in order to operate at the transition the parameters of the model must satisfy the relation  $rK = a_p$ , where  $a_p$  as a function of  $p/(1-p)$  is a stretched exponential function.

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